Optimum Filtering and Smoothing of Buoy Wave Data

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The problem of estimating ocean wave heights given accelerometer and pressure measurements from a spar buoy is described. A Kalman optimum filter and a smoother are used to combine these measurements to estimate four state variables: wave height and its time integral, and buoy vertical displacement and velocity. The time history of the ocean waves is presented as a second-order system driven by white noise. The buoy dynamics in the vertical are represented as a second-order system driven by the waves with additional white noise. Numerical computations were carried out for the state vector of order four using five minutes of digitally recorded data from an actual spar buoy. One of the state variables, buoy vertical displacement, is compared directly with the accelerometer output as doubly integrated by another numerical technique and found to be slightly more stable. Inspection of the measurement residuals from the filter and smoother suggest that the model or parameters chosen are not in final form.

Introduction

IN this paper the state vector approach and optimum estimation theory are used to estimate the height of ocean waves. The impetus for this problem came from difficulties encountered in analyzing wave and motion data from Flordia State University's 100-ft buoy TRITON.[1] This buoy was primarily intended as a platform for meteorological instruments, but as a side project it was equipped with resistance wire gages. Early attempts at numerical double integration of the vertical accelerometer output diverged rapidly.

While studying the buoy motion problem the author investigated a work[2] which applied the Kalman filter to the study of ocean current meter dynamics, but was not able to pursue the problem further at that time. The work described here was begun two years later using data from a smaller spar buoy, Lockheed's Measurement and Comparison System (MCS).[3]

The particular problem considered was to compute a wave spectrum or time history from digitized records of relative wave height and apparent buoy vertical acceleration. Since the vertical motion of the buoy was a significant fraction of the wave height over much of the frequency range, it was not possible to combine the low frequency part from the accelerometer and the high frequency part from the wave gage. Furthermore it would be difficult to combine spectral components at each frequency because of unknown phase relations. Thus, it was necessary first to work in the time domain and construct a history of the ocean surface, which requires numerical double integration of the vertical accelerometer output. At least three factors make this numerical integration difficult: unknown initial conditions, nonverticality of the accelerometer, and the finite quantization step in the digitization. An ordinary numerical quadrature leads to rapid divergence.

In the work reported here the problem was solved by the Kalman filter, which has been successfully applied to such areas as inertial navigation and orbit estimation. The results of its application to a single set of data from the MCS[3] buoy are given below and possible future extensions of the method are outlined. Because the Kalman filter is a time domain technique, no spectral calculations are presented. The remainder of this paper contains additional background on measurements from buoys and the Kalman filter.

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Measurements from Buoys

Time histories of ocean surface elevations fixed locations can be used to compute nondirectional wave spectra which are required to test theories of the generation, propagation, and decay of ocean waves. The wave record itself may be of interest in such applications as determining the highest wave encountered. Over the last eighteen years most such data taken in the ocean has been recorded by shipborne Tucker[4] wave recorders. This system measures subsurface pressure through the ship's hull and vertical acceleration of the same part of the ship by a damped, gimballed accelerometer. The analogue voltage from the accelerometer is twice integrated electronically, added to the voltage from the pressure gage, and the results recorded on a strip chart.

Wave measuring buoys have been successfully constructed which follow the sea surface and are equipped with radio transmitter and a "vertical" accelerometer, either strapped down to the buoy[5] (imballed to remain vertical.[6] The first type is much simpler mechanically, but since the accelerometer does not remain truly vertical its output cannot be doubly integrated. Instead, a direct spectrum of acceleration can be computed and multiplied by a suitable frequency factor to convert it to a spectrum of surface height. Tucker[7] has analyzed the errors due to tilting to be expected in the resulting spectrum and has concluded that the worst errors occur at lower frequencies than are normally encountered.

Today increasing use is being made of moored or drifting data buoys because of ship costs and the desire for long time series of geophysical data. Usually they are instrumented for multiple measurements. The author has analyzed data from the TRITON[1] buoy and from Lockheed's MCS[3] buoy. Both are spar buoys constructed as semistable platforms for meteorological measurements over the open ocean. Wave measurements were a secondary consideration, and TRITON was equipped with resistance wires and the MCS with a subsurface pressure gage for this purpose. For simplicity and economy their vertical motions were monitored by strapped-down vertical accelerometers; this type of buoy remains within 10°-15° of vertical under normal conditions so the accelerometer output is a reasonable facsimile of buoy vertical acceleration. All data were digitized several times per second and recorded digitally. As listed by Smith and Perry[8] digital data techniques offer a number of advantages for oceanographic data including simpler tape decks, direct computer compatability, stability, and compatibility with multiple inputs. It was desired to minimize on-board signal processing, putting the burden on a fast digital computer which analyzes the data later.

Estimation Model

One of the main advantages of the Kalman filter is that it can easily be computer-coded for processing measurements and estimating the state of a system. The estimate weights all information about the quality of the measurements and the dynamics of the system and can be shown to be optimal for a linear (or linearized) system under a variety of conditions. For a general account and more detail on the mathematical notation employed, the reader is referred to Kalman,[9] Meditch,[10] Leondes,[11] or Jazwinski.[12] Here the specific problem will be cast in the required form using state variable notation and general Kalman solution and its properties will be stated without proof. First the dynamics of the buoy and the wave spectrum must be modeled as differential equations driven by white gaussian noise, then collected as single first-order matrix equation and converted to a matrix finite difference equation.

Ocean waves are a stochastic process with a fairly narrow band of frequencies, rather than a white noise process. A typical autocorrelation function for ocean waves decays after several oscillations at the dominant frequency. Latta and Bailie[13] have shown that both the well-known Neumann[14] spectrum and the Pierson-Moskowitz[15] spectrum have autocorrelation functions with decaying oscillatory terms, although not of truly constant period. A process with an exponential cosine autocorrelation can thus be a reasonable model for ocean waves, and such a process can be modeled as a second-order differential equation driven by white noise.

Consider the second-order system specified by the two state variables x_1 and x_2 with

$$\dot{x}_1 + 2\alpha x_1 + (\alpha^2+\beta^2)x_2 = w_1 \qquad (1)$$

$$\dot{x}_2 = x_1 \qquad (2)$$

where w_1 is a white noise process. According to Papoulis,[16] and Bryson and Ho,[17] both x_1 and x_2 have exponential cosine autocorrelation functions, of the respective forms

$$\phi_{11}(\tau) \sim \exp(-\alpha|\tau|)[\cos\beta|\tau| - \frac{\alpha}{\beta}\ 3|\tau|] \qquad (3)$$

and

$$\phi_{22}(\tau) \sim \exp(-\alpha|\tau|)[\cos\beta|\tau| + \frac{\alpha}{\beta}\sin\beta|\tau|] \qquad (4)$$

The constants in front of each expression have been omitted; the variances of the driving noise process w_1 can always be adjusted as needed. According to Papoulis,[16] x_2 has power spectral density proportional to

$$S_2(w) \sim \frac{1}{(w^2-\alpha^2-\beta^2)^2+4\alpha^2w^2} \qquad (5)$$

Consequently, x_1 the time derivative of x_2, has a power spectral density proportional to

$$S_1(w) \sim \frac{w^2}{(w^2-\alpha^2-\beta^2)^2+4\alpha^2w^2} \qquad (6)$$

Initially x_2 was chosen as wave height and x_1 as its derivative (wave speed); later it was decided to make x_1 wave height and x_2 as its time integral, so that the wave height would have a power spectral density given by Eq. (6) instead of Eq. (5), since Eq. (6) has zero power at zero frequency. Figure 1 shows an averaged hindcase spectrum and an averaged idealized spectrum taken from a paper[18] read at the Ninth Symposium

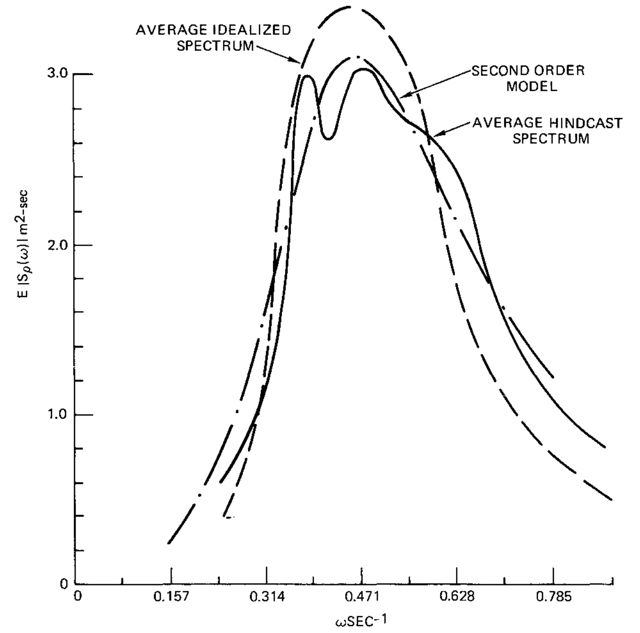

Fig. 1 Ocean wave spectrum from second-order model compared with average hindcast spectrum and average idealized spectrum.

on Naval Hydrodynamics. Superimposed on these is a curve generated by adjusting α and β in Eq. (6) to show that this model can give a reasonable spectrum. The spectral density [Eq. (6)] behaves like w^{-2} for large frequency and like w^2 for small frequency. Although the overall spectral shape seems reasonable according to Fig. (1), the results might not be acceptable if a narrow portion of the spectrum were being studied (such as very low frequency). The order of the wave model of Eqs. (1) and (2) might then have to be changed, depending on the purpose and subsequent analysis of the data.

The vertical dynamics of the cylindrical spar buoy can be modeled as a nearly critically damped harmonic oscillator (Fig. 2) with a restoring force proportional to the difference between the buoy's displacement from equilibrium h and the wave displacement from equilibrium y. Using the state variables

$$x_3 = h \qquad (7)$$

and

$$x_1 = y \qquad (8)$$

we have the equations

$$\dot{x}_3 = x_4 \qquad (9)$$

$$\dot{x}_4 = -2\omega\zeta x_4 - \omega^2(x_3 - x_1) + w_4 \qquad (10)$$

Here ζ is the damping coefficient ($\zeta = 1$ for critical damping) and ω is the angular natural frequency of heave, with

$$\omega^2 = \frac{g}{L} \qquad (11)$$

for a cylindrical spar of draft L. As explained in the following, to account for the added mass of water entrained by the base plate, an L larger than the actual draft was used. According to Eq. (10), the buoy is driven by (x_3-x_1), the difference between wave height and the vertical displacement of the buoy. The inclusion of the white noise process w_4 is intended to cover things neglected in the model, such as pitch or roll indicated by ψ in Fig. 2. As previously explained, an earlier version used x_2 as wave height and x_1 as wave speed,

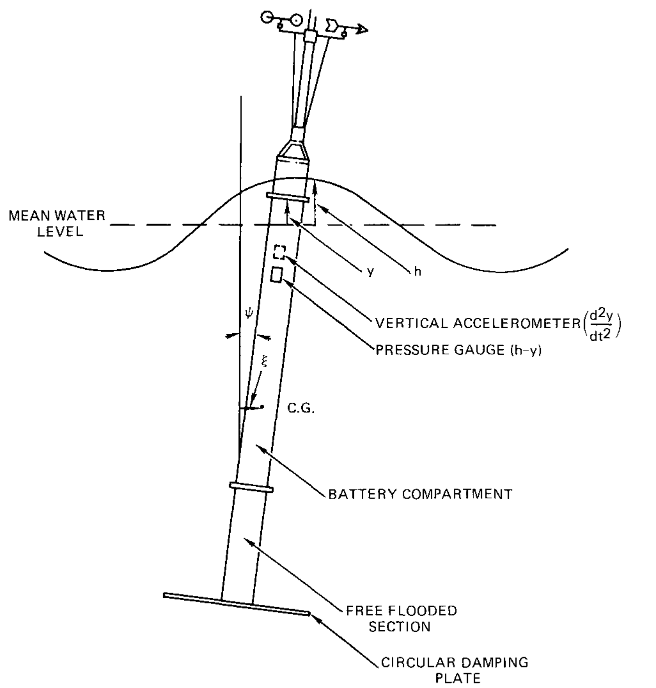

Fig. 2 Coordinate system used to model buoy dynamics.

but the model presented here does not include wave speeds as a state variable and so assumes damping proportional to buoy speed only. This seemed to work better than the earlier version and is partly justified by the fact that most of the vertical damping takes place at the large base plate 28 ft down where wave orbital motion is partly attenuated.

The state vector for the problem is defined as the column matrix of the four state variables. Then the four first-order differential equations can be summarized as one first-order matrix differential equation

$$d/dt \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2\alpha & -(\alpha^2+\beta^2) & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 & 0 & -\omega^2 & -2\omega\zeta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} W_1 \\ 0 \\ 0 \\ W_4 \end{bmatrix} \quad (12)$$

which can be written in the standard form

$$\frac{dx}{dt} = Fx + w \quad (13)$$

F is called the dynamics matrix.

The two components of the measurement vector z, buoy vertical acceleration z_1, and wave pressure converted to depth of water z_2, are related to the state vector by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 & -\omega^2 & -2\omega\zeta \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (14)$$

or in standard form

$$Z = Hx + v \quad (15)$$

where H is the measurements matrix. By definition v_1 and v_2 are white noise sequences. The main error relating pressure readings to wave height is the attenuation of wave pressure with depth—the gage was located six ft below mean water level. This effect discriminates against high frequencies and is not adequately modeled as white noise. Neither is the accelerometer's main error source, failure to remain vertical due to buoy pitch and roll, adequately modeled as white noise. A more complete model would include pitch and roll as state variables. Equation (14) would be simpler if vertical acceleration were included as a state variable but then Eq. (12) would contain a third-order differential equation.

Estimation Calculations

For the case of constant F matrix, it can be shown (Meditch,[10] p. 32) that the solution to Eq. (13) is

$$x(t) = e^{Ft}x(o) + \int_o^t e^{F(t-\tau)} w(\tau)d\tau \quad (16)$$

defining a matrix exponential by an infinite series. $\phi(t) = e^{Ft}$ is known as the state transition matrix. To convert Eq. (16) to a finite difference equation it is convenient to write the solution of Eq. (16) for the interval Δ between data samples. The state transition matrix for the problem is

$$\phi(\Delta) = e^{F\Delta} = I + F\Delta + \frac{F^2\Delta^2}{2!} + \ldots + \frac{F^N\Delta^N}{N!} + \ldots \quad (17)$$

It is possible to derive an exact state transition matrix ϕ from a dynamics matrix F, but it was found much simpler to use Eq. (17) numerically and truncate the series after the terms became small enough. With the state transition matrix in Eq. (17), Eqs. (13) and (15) can be written

$$x(k+1) = \phi\, x(k) + w(k+1) \quad (18)$$

$$Z(k+1) = H\, x(k+1) + v(k+1) \quad (19)$$

using the discrete index k to label the time $k\Delta$. $w(k+1)$ and $v(k+1)$ are now discrete white gaussian sequences. No attempt is made to derive these from Eq. (16); only their statistical properties are known. The fact that they are white or uncorrelated sequences is expressed by their convariance matrices as

$$E[w(k)w^T(j)] = Q\, \delta_{jk} \quad (20)$$

$$E[v(k)v^T(j)] = R\, \delta_{jk} \quad (21)$$

where the operator E denotes a mathematical expectation value, k and j are indices labeling discrete times, δ is the Kroniker delta, and T denotes a matrix transpose. Equations (20) and (21) are square matrices since they are the product of a column vector and a row vector.

When the problem has been formulated as in Eqs. (18-21) the Kalman optimum solution can be given. $x(k+1)$ in Eq. (18) denotes the actual state vector at $k+1$, whereas the best that can be known, because of system and measurement noise, is an optimum estimate of the state vector. The notation $\hat{x}(k/k)$ means the optimum estimate of the state vector at time k based on all informtion up to k, while

$$\hat{x}(k+1/k) = \phi\hat{x}(k/k) \quad (22)$$

represents this estimate extrapolated one time step forward and can be shown to be the optimum estimate at time $(k+1)$ based on all information up to time k.

The Kalman solution gives an estimate that is a linear combination of the measurements. It can be shown to give an estimate that is unbiased, i.e.,

$$E[\hat{x}(k/j)] = E[x(k/j)] \tag{23}$$

The error can also be shown to have minimum variance, i.e.,

$$E\{[x(k)-\hat{x}(k/k)]^T[x(k)-\hat{x}(k/k)]\} \tag{24}$$

is a minimum. Thus the estimate is a best fit in an average, least-squares sense. Other assumptions are that $\hat{x}(k/j)$ and

$$[\hat{x}(k/j)-x(k)]$$

are Gaussian random vectors for all k and j and that the error vector is independent of the estimate, i.e., the matrix

$$E\{[x(k)-\hat{x}(k/k)]\hat{x}^T(k/k)\} = 0 \tag{25}$$

The mean square error, Eq. (24), is a scalar, whereas the covariance of the error

$$P(k/k) = E\{[\hat{x}(k/k)-x(k)][\hat{x}(k/k)-x(k)]^T\} \tag{26}$$

is a square matrix that is generated as a by-product of the Kalman solution. A part of the Kalman solution is the recursive generation of the covariance matrix by the three matrix equations

$$P(k+1/k) = \phi p(k/k)\phi^T + Q \tag{27}$$

$$B(k+1) = p(k+1/k)H^T[H\,p(k+1/k)H^T+R]^{-1} \tag{28}$$

$$P(k+1/k+1) = [I-B(k+1)H]P(k+1/k) \tag{29}$$

From these equations the entire time history of the error covariance matrix can be generated without the use of any data at all. Many published applications consist of just such a covariance study.

The B matrix in Eq. (28) is the Kalman gain matrix. The Kalman filter operates in a "predict-correct" fashion; after the extrapolation Eq. (25), the predicted state vector is used to predict a measurement vector by Eq. (19). This is substracted from the actual measurement vector when it arrives to produce an important measure of filter performance called the residuals sequence;

$$r(k+1) = z(k+1) - H\hat{x}(k+1/k)$$

$$= z(k+1) - H\phi\hat{x}(k/k) \tag{30}$$

Finally, an optimum estimate of the state at time $(k+1)$ based on all information up to time $(k+1)$ is computed as the predicted state vector plus a correction given by the residual Eq. (30) multiplied by the Kalman gain matrix [Eq. (28)]:

$$\hat{x}(k+1/k+1) = \hat{x}(k+1/k) + B(k+1)r(k+1)$$

$$= \hat{x}(k+1/k) + B(k+1)[Z(k+1) - H\phi\hat{x}(k/k)] \tag{31}$$

This completes the solution.

Some properties of the Kalman filter follow from Eqs. (27-31). Not only does it give an optimum estimate of all state variables at once, using all available information, but it can estimate more state variables than were measured. Especially important in real time applications is its recursive property; it processes only the most recent set of measurements and the previous state vector at each step. As by-products is provides the covariance matrix as a measure of fundamental model accuracy and the residuals sequence as a measure of the model's appropriateness to actual data.

After the forward Kalman filter has been implemented to the end of the data set (n), the state vector estimates can be further improved by an optimum smoother which works backward over the data starting with $P(n/n)$ and $\hat{x}(n/n)$. The smoother used here is due to Rauch et al.[19]; other smoothers have been given by Fraser[20] and by Bryson and Frazier.[21] The optimum smoothed estimate $\hat{x}(k/n)$ at time k based on all information up to n, and the error covariance $P(k/n)$, are given by

$$\hat{x}(k/n) = \hat{x}(k/k) + C(k)[\hat{x}(k+1/n) - \phi\hat{x}(k/k)] \tag{32}$$

$$P(k/n) = P(k/k) + C(k)[P(k+1/n) - P(k+1/k)]C^T(k) \tag{33}$$

Similar to Eq. (31), the smoothed estimate consists of the unsmoothed estimate plus a weighted difference between the next smoothed estimate and the extrapolated estimate. The smoothing matrix $C(k)$ is given by

$$C(k) = P(k/k)\phi^T P(k+1/k)^{-1} \tag{34}$$

It would be wasteful to store all the $P(k/k)$ and $P(k+1/k)$; they can be generated recursively backwards using formulas given by Rauch et al.[19] In the data set studied in the next section it was noted that after the forward filter had processed about 5% of the data the elements of the P matrix has reached nearly constant values. Thus only Eqs. (32) and (33) were used and the smoother used the constant P values over only the last 95% of the data.

Sample Numerical Results

All computations were done on an IBM 360 computer, with a CALCOMP plotter available for visual inspection of results. Although several models were tried, the one specified by Eqs. (13) and (15) was found to work best. A general subroutine which implements the Kalman filter, Eqs. (25-29) was supplied to the author by H. E. Rauch. It was initialized by reading from data cards the dimensions of the state and measurement vectors. This sets the dimensions of the ϕ, $P, H, Q,$ and R matrices, whose elements were then read in. The data were then read into arrays and the filter implemented starting with all state variables at zero.

Equations (13) and (18) were used to set the state transition matrix, truncating the series after the first eight terms. The wave parameters $\alpha = 0.08$ and $\beta = 0.45$ were chosen by inspection of the auto-correlation function of the pressure record. The results were found to be not very sensitive to changes in these values. The buoy parameters used were $\zeta = 0.8$, $\omega = 0.61$. This ω was intended to take into account the added water mass entrained by the base plate; this combination of ζ and ω gives a heave response function that is a good approximation to one measured for a scale model of the buoy in a wave tank. The elements of the Q and R matrices were varied over a wide range; the results presented below used $Q(1) = 1.0$, $Q(4.4) = 10.0$, $R(1,1) = 0.1$, $R(2,2) = 1.0$ and all other elements of both matrices were zero. The error covariance matrix P was assigned large initial diagonal values and zero off-diagonal values. The steady values reached by the P elements seem to be independent of the initial values so long as these are large. All diagonal elements of the measurements and will cause the filter to follow the data only, ignoring the model. Models with a state vector of order five, including acceleration, were also tried.

The only data set used consisted of a 5-min time series of simultaneous acceleration and pressure data taken by the MCS buoy[3] in 60 ft of water under light swell conditions. The sampling interval was 0.5 sec. After conversion to cm of water, the pressure record has a variance of 662.0 cm^2 while the acceleration record has a variance of 53.8 $cm^2\ sec^{-4}$. The correlation between the two records is 0.63 and increases to

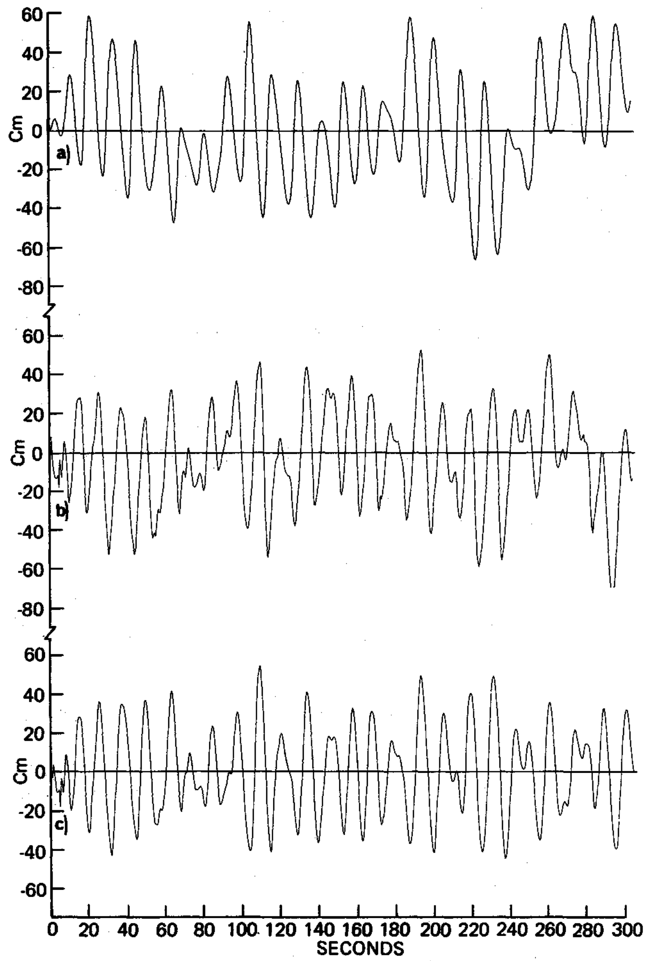

Fig. 3 a) Vertical accelerometer data doubly integrated by modified trapezoidal rule. b) Buoy vertical displacement estimated by forward Kalman filter. c) Buoy vertical displacement estimated by forward filter plus smoother.

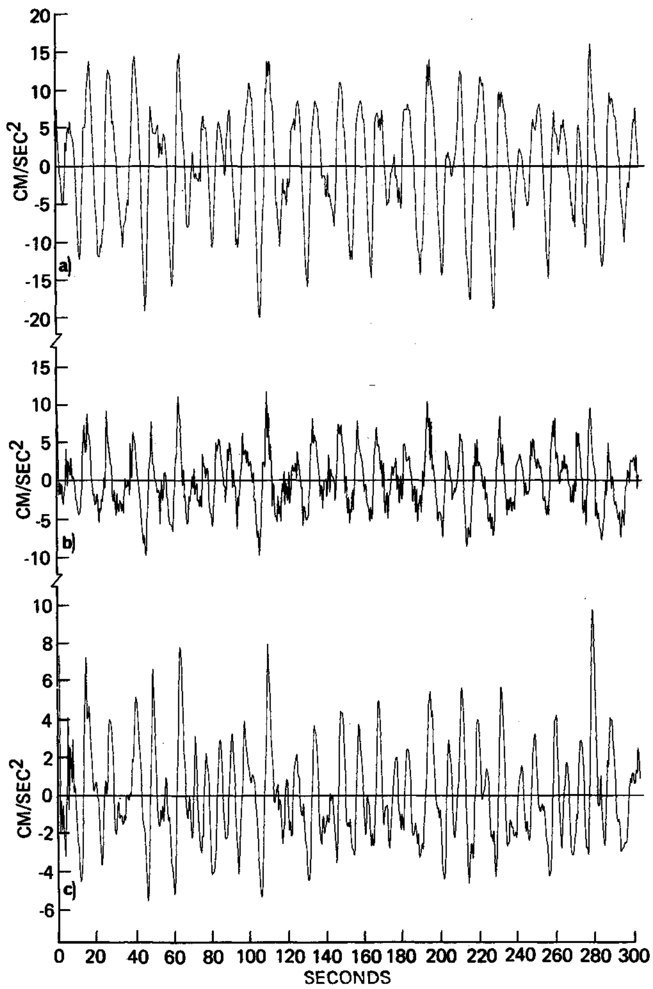

Fig. 4 a) Vertical accelerometer data. b) Acceleration residual after forward filter. c) Acceleration residual after forward filter plus backward smoother (note scale).

0.85 if the acceleration is lagged behind the pressure record by 2 data points.

Since real data were available more emphasis was put on the residuals sequences than on the elements of the convariance matrix. In some cases when buoy acceleration was included as a state variable the diagonal element of the P matrix corresponding to displacement did not settle down but increased indefinitely in time. This was one reason for not using such models.

Two criteria were used for evaluating results. The first was comparison of x_3, the estimated displacement, with the double integration by a method which successfully controls the divergence using an exponential factor in the integrand; thus, integrating $a(t)$ to get $v(t)$ one can use

$$v(t)=\int_o^t a(\tau)e^{-\delta(t-\tau)}d\tau \tag{35}$$

choosing δ to be as small as possible while still controlling divergence. A trapezoidal rule approximation to this integral has been tested with data created in the lab. There is a tradeoff between amplitude reduction and elimination of trends (divergence control); complete suppression of trends may reduce the amplitude 10% to 20% after two integrations. Figure 3 compares such a double integration (Fig. 3a) with the same quantity estimated by the forward Kalman filter (Fig. 3b) and by the forward filter plus the optimum smoother (Fig. 3c). Any trends longer than 15-20 sec are not believed to be real. Figure 3a has such trends while Fig. 3b has fewer trends but more high frequency spikes; thus neither is clearly superior. The curve in Fig. 3c can be said to be superior because of the complete absence of long-term trends.

If the model is correct the measurement residuals will form an uncorrelated white noise sequence and will not be correlated with the measurement sequence. Figure 4a shows the acceleration data, Fig. 4b, the acceleration residual and after, the forward filter and Fig. 4c, the residual after the filter and smoother. Some correlation between residuals and data is evident in both cases. (Figure 3c has different scale). The residual after filtering has a variance 26.5% of that of the raw data, and after smoothing the figure is 11.9%. Figure 5 shows the same quantities for the pressure. The residuals look uncorrelated, but so does the pressure data. After the forward filter the variance is 32.4% of the pressure and after the smoother it is 15%. These numbers suggest the acceleration data was slightly better tracked than the pressure data, possibly indicating the buoy motion part of the model is better than the wave model.

Clearly the model is not yet in final form and more work is needed and more data sets under varying sea conditions. The wave estimates are now shown because they cannot be directly compared with any other quantity. While inspection of the residuals is useful in evaluating a model, it cannot be solely relied upon. Thus one model included acceleration as a state vector and represented the buoy dynamics by a third-order differential equation; the residuals had small variance but the buoy displacement had a large and physically impossible trend. A certain amount of trial and error are indespensible since changes in the buoy, wave, and noise parameters are not orthogonal.

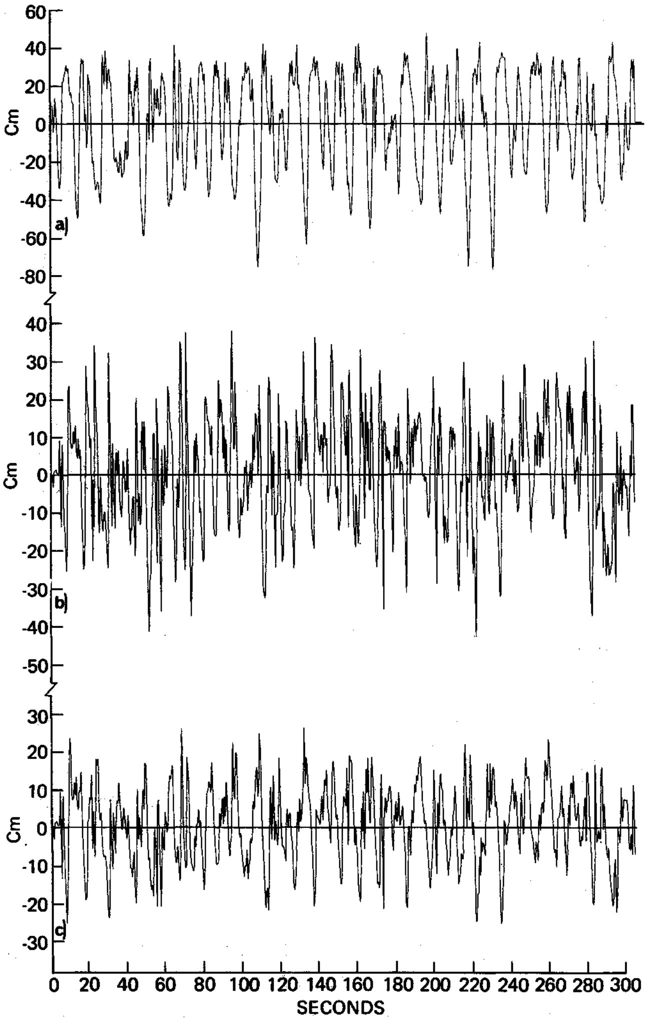

Fig. 5 a) Pressure data (note scale). b) Pressure residual after forward filter. c) Pressure residual after forward filter plus smoother.

Conclusions

The measurement problem has two aspects: to measure ocean waves from a moving buoy using a nongimballed (strapped-down) accelerometer and a relative wave gage, and to record sensor outputs digitally without any analog (electronic) operations unless necessary to prevent aliasing. The Kalman filter then offers the theoretically optimum way to reduce such data. The computer can simulate any analog operation, but after such an operation the original is lost. Kinsman[22] holds the opinion that it is irresponsible to physically filter geophysical records at the time they are taken since later someone may wish to re-work the data from a different point of view.

Thus far this method has been applied only to a single data set taken in relatively small waves. The forward filter is not yet final but after the smoother, the results are slightly better than the result of a seemingly simpler double integration and combination with the pressure record. The Kalman filter does more arithmetic than the simpler method, but on a fast digital computer the difference would hardly be noticeable as long as the measurement vector were not of too high an order. Considerable effort was expended to find a workable Kalman model for this problem. However, once a model was established, routine running would be easy, and would offer the advantage over the simpler method of estimating all state variables in a single pass over the data. The smoother requires a second pass backwards.

The fundamental advantage of the Kalman filter is its capability of extension to more complicated models with more measurements and state variables. The simpler method works for this case only (one-dimensional problem, spar buoy which remains nearly vertical). Specifically the Kalman filter might be extended to process measurements from several strapped-down inertial sensors[2] and a compass, using these to estimate wave slopes (for directional spectra) as well as heights. It might also be extended to buoy shapes other than a spar, making it possible to estimate waves from any buoy using strapped-down interial sensors and a pressure or other wave gage.

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